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Rainbow graph splitting

A. Monti, B. Sinaimeri *

Department of Computer Science, "Sapienza" University of Rome, Via Salaria 113, 00198 Roma, Italy

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ABSTRACT

Given an integer c, an edge colored graph G is said to be rainbow c-splittable if it can be decomposed into at most c vertex-disjoint monochromatic induced subgraphs of distinct colors. We provide a polynomial-time algorithm for deciding whether an edge-colored complete graph is rainbow c-splittable. For not necessarily complete graphs, we show that the problem is polynomial if $c = 2$, whereas for $c \geq 3$ it is NP-complete even if the graph has maximum degree $2c - 1$. Finally, it remains NP-complete even for 2-edge colored graphs of maximum degree $7c - 14$.

1. Introduction

The problem. Throughout this paper graphs are always simple, i.e. without loops or multiple edges. As usual K_n will denote the complete graph on n vertices. Let $G = (V, E)$ be an edge-colored (not necessary properly) connected graph. Then, G is called *monochromatic* if all its edges are of the same color. In this paper coloring would always refer to an edge-coloring except when indicated otherwise. We say a graph is *r-colored* if at most r colors appear on its edges. For $V' \subseteq V$ we denote by $G[V']$ the graph induced by V'. We denote by $[n]$ the set $\{1, \ldots, n\}$. For undefined terms and concepts the reader is referred to [2].

Fixed an integer c, an edge-colored graph $G = (V, E)$ is said to be rainbow c-splittable if there exists a decomposition of G into at most c monochromatic vertex-disjoint induced subgraphs of pairwise distinct colors (hence the name rainbow splitting). The partition $V_1, \ldots, V_{c'}$ of V (with $c' \leq c$) that generates such a decomposition is referred as a rainbow c-split of G. We allow one or more sets of the partition to be of cardinality one. Indeed, in this case the induced subgraph has no edges, so we consider it being of an arbitrary color. In this paper we study the computational complexity of the following problem:

Rainbow c-Splitting Problem
Instance: An edge-colored graph G.
Question: Is G rainbow c-splittable?

Note that we deal only with the cases where c is a fixed integer (meaning that c is not part of the input).

It is worth observing that the rainbow c-splitting problem can be also considered as a relaxed vertex coloring problem. Given an edge-colored graph, is there a coloring of the vertices with at most c colors such that two vertices can be of the same color a only if they are either not adjacent or connected by an edge of color a?

Our results. It is clear that an edge-colored graph is rainbow 1-splittable if and only it is monochromatic. Thus the first non trivial case of the Rainbow c-Splitting problem, is for $c = 2$. We show that rainbow 2-splittable graphs can be

* Corresponding author.
E-mail addresses: monti@di.uniroma1.it (A. Monti), sinaimeri@di.uniroma1.it (B. Sinaimeri).

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recognized in polynomial time by providing a reduction to the well-known 2-SAT problem [1,5]. Then we investigate the class of complete graphs. We provide a polynomial-time algorithm that decides whether an edge colored complete graph K_n is rainbow c-splittable. For not necessarily complete graphs, we show that the Rainbow c-Splitting problem is NP-complete for $c \geq 3$ and remains so even if the graph is c-colored and has maximum degree $2c - 1$. Furthermore, it is still difficult if we limit the number of colors on the edges of the graph. In particular, the problem is NP-complete even for 2-colored graphs of maximum degree $7c - 14$.

Related works. Graph partitioning problems are extensively studied in the literature in many aspects and variations. The well-known vertex coloring problem fits itself in this framework: Partition the vertex set into the minimum number of subsets each of which induces a stable in the original graph. Several variations of this problem have been introduced, leading to interesting new concepts and challenging problems [3,7,16,19].

Some graph partitioning problems require that the partition satisfies a color pattern. In particular, in [13] the authors considered the problem of determining the minimum number of subsets in which the vertex set of a colored graph can be partitioned such that each of the subsets induces a monochromatic clique proving that it is NP-hard. Observe that the Rainbow c-Splitting problem concerning complete graphs not only requires a partition into monochromatic cliques, but also a rainbow partition, i.e. all the cliques must be of pairwise distinct colors.

In the same flavor is the problem of $\mathbf{a}$-split colorings [11] which comes as a generalization of the well-known problem of recognition of *split graphs* [9]. Fix an integer c, let $\mathbf{a} = (a_1, \ldots a_c)$ be a vector of nonnegative integers. A c-*coloring* of the edges of a complete graph is called $\mathbf{a}$-split if there is a partition of the vertex set V into c subsets such that for any $i \in [c]$ every set of $a_i + 1$ vertices in the class V_i contains an edge of color i. Clearly, for c-colored complete graphs the $(1, 1, \ldots, 1)$-splitting and the rainbow c-splitting are equivalent properties. However, observe that this kind of connection cannot be extended to complete graphs with more than c colors on their edges or to graphs that are not complete. For $c = 2$ there are well-known simple algorithms that recognize $(1, 1)$-split complete graphs in polynomial time and the same result holds also for arbitrary graphs (see for example [10]). Concerning the case $c \geq 3$, the authors of [11] proved that the family of $\mathbf{a}$-split complete graphs can be characterized by a finite list of forbidden induced colored subgraphs. This implies that once such a list is obtained, it would be possible to decide in polynomial time the rainbow c-splitting for c-colored complete graphs. However, an explicit description of such a list seems very difficult and it is still unknown even for the case $c = 3$ (for the case $c = 2$ see [9]). Our result implies the first *explicit* polynomial-time algorithm for the $(1, 1, \ldots, 1)$-splitting problem. Moreover the algorithm is not based on forbidden induced colored subgraphs.

Finally, it is worth mentioning the connection with the *adaptable coloring* problem (see for example [6,12,15]). Given an edge colored graph an adaptable coloring is a vertex coloring such that there is no edge $e = \{u, v\}$ with $c(u) = c(v) = c(e)$. Considering the rainbow splitting problem as a vertex coloring problem, it is clear that these two problems impose opposite requirements. They also present different kind of difficulties. Indeed, in [6] it is proved that deciding the existence of an adaptable coloring with c colors for a c-colored graph is NP-complete for $c \geq 4$ and for complete graphs. Moreover, for $c = 3$ the complexity is still unknown even when restricting attention to complete graphs. However, in [8] some evidence is given that this problem is not NP-complete. Despite their different nature in the case of graphs, their generalizations to multigraphs define basically the same problem. Indeed, consider a multigraph where multiple colored edges appear between two vertices. It is clear that for two vertices u, v allowing $c(u)$ and $c(v)$ to be of the same color a only if there is some edge of color a connecting them (rainbow splitting), or allowing it if none of the edges is of color a (adaptable coloring), are mainly the same type of requirements. They both allow $c(u) = c(v) = a$ only if a belongs to some set of *permissible* colors. In the first problem this set is C_{uv}, where C_{uv} denotes the set of colors appearing on the edges connecting u and v, in the second the set is $C \setminus C_{uv}$, where C is the set of colors appearing on all the edges of the multigraph.

This kind of requirement provides natural interpretations for various problems; as matrix partitions of graphs [7,8,12], full constraints satisfaction problems [6] or scheduling problems [15].

To get some intuition concerning rainbow splitting in case of graphs consider the following simple scenario: Suppose there are n games to be scheduled in time. The constraint is that two games can be played simultaneously only on a particular day. Now, given a fixed integer c, can we schedule the games such that all of them are played in a feasible way, within c days? It should be clear that the Rainbow c-Splitting problem models this situation.

2. Rainbow 2-Splitting

In this section we deal with the case $c = 2$. We prove that the Rainbow 2-Splitting problem can be solved in polynomial time. For 2-colored graphs, in [10] it is proved that the problem can be solved in linear time. Here we extend this result to edge-colored graphs with an arbitrary number of colors. In order to simplify the forthcoming proofs we define the rainbow splitting problem in correspondence to a set of colors. Given a set of colors A we say that a graph G is rainbow A-splittable if it can be decomposed into monochromatic induced subgraphs of pairwise distinct colors from A. Clearly a graph is rainbow c-splittable if and only if there is a set A of at most c colors for which G is rainbow A-splittable. The proof will follow straightforwardly by the next lemma.

Lemma 1. *Given an edge-colored graph G on n vertices and two colors* a, b, *there is an algorithm that decides in time* $O(n^2)$ *if the graph G is rainbow* $\{a, b\}$*-splitting.*

Proof. We prove the lemma by showing that the problem is reducible to the well-known problem 2-SAT where, given a formula F in conjunctive normal form where each clause has two literals, we ask if there is some assignment of the literals that makes the formula F true. It is known that 2-SAT can be solved in polynomial time [1,5]. Given an edge colored graph $G = (V, E)$ and the colors a, b we construct a formula F as follows: To each vertex $v \in V$ and edge $e \in E$ of color different from a and b the Boolean variables x_v and x_e correspond respectively; to every edge uv of color a (b) corresponds the clause $x_u \vee x_v$ ($\neg x_u \vee \neg x_v$) and to every edge uv of color different from u and v correspond the following three clauses: $\neg x_u \vee \neg x_v$, $x_u \vee x_{uv}$ and $x_v \vee \neg x_{uv}$. Now, if F is satisfiable consider an assignment that makes F true and let V_a and V_b be the set of vertices whose corresponding variables are respectively true and false. It is not difficult to see that the partition (V_a, V_b) is a rainbow $\{a, b\}$-split in G. Indeed, they trivially partition the vertex set and every edge $uv \in G[V_a]$ must be of color a otherwise the clause $\neg x_u \vee \neg x_v$ cannot be satisfied as x_u, x_v are both set to true. Similarly, every edge $uv \in G[V_b]$ must be of color b otherwise at least one from the clause $x_u \vee x_v$, $x_u \vee x_{uv}$ and $x_v \vee \neg x_{uv}$ cannot be satisfied as (x_u, x_v are both set to false). Conversely, let (V_a, V_b) be the rainbow $\{a, b\}$-split of G. Assign true (false) to each variable whose corresponding vertex belongs to V_a (V_b) and assign true to each variable x_{uv} whose corresponding edge uv does not belong in $G[V_a]$. It is simple to verify that this assignment makes F true. This concludes the proof. □

At this point given a r-colored graph G on n vertices, we can decide rainbow 2-splitting trivially considering all the possible $\binom{r}{2}$ pairs of colors. Thus, we can state the following:

Theorem 1. *Given an r-colored graph G on n vertices, there is an algorithm that solves the Rainbow 2-Splitting problem in time $O(r^2 n^2)$.*

3. Rainbow c-Splitting of complete graphs

The main purpose of this section is to provide a polynomial-time algorithm that decides if a given edge-colored complete graph K_n is A-splittable for a given set A of c colors. Without loosing generality we can suppose from now on that the set of colors A is the set $[c]$. It should be clear that if we have such a polynomial-time algorithm then we can decide rainbow c-splitting trivially in polynomial time by considering all the possible $\binom{r}{c}$ sets of c colors, where r is the number of colors appearing on the edges of the graph K_n.

We begin by proving a technical lemma stating some properties related to the maximum monochromatic clique in rainbow $[c]$-splittable complete graphs.

Lemma 2. *Let K be a complete edge-colored graph with $V(K) = V$. For any color $a \in [c]$ let C be the maximum monochromatic clique of color a in K. Then K is rainbow $[c]$-splittable if and only if there are two sets of vertices X and Y where $X \subseteq C$ with $|X| < c$ and $Y \subseteq V \setminus C$ with $|Y| \leq |X|$ such that for the set $Z = (C \setminus X) \cup Y$ both of the followings hold*

- *$K[Z]$ is a monochromatic clique of color a*
- *$K[V \setminus Z]$ is rainbow $[c]$-splittable with $V_a = \emptyset$ (i.e. there is no monochromatic clique of color a).*

Proof. The only if part is trivial as if $V_1, \ldots, V_c$ is the rainbow $[c]$-split of $K[V \setminus Z]$ and $V_a = \emptyset$ then setting $V_a = Z$ we obtain a $[c]$-split for K.
Concerning the if part let C be the maximum monochromatic clique of color a in K and let $V_1, \ldots, V_a, \ldots, V_c$ be a $[c]$-split of K. Let $X = C \setminus V_a$ and $Y = V_a \setminus C$. Observe that $|X| \leq c - 1$ as otherwise we will have an edge of color a in some set V_i with $i \neq a$. Furthermore $|Y| \leq |X|$. Indeed $|V_a| = |V_a \setminus C| + |V_a \cap C| = |Y| + |V_a \cap C|$ on the other hand $|C| = |C \setminus V_a| + |V_a \cap C| = |X| + |V_a \cap C|$ and as $|V_a| \leq |C|$ the result follows. Now, let $Z = (C \setminus X) \cup Y = (V_a \cap C) \cup (V_a \setminus C) = V_a$ and clearly setting $V_a = \emptyset$ we obtain a rainbow $[c]$-split for $K[V \setminus Z]$ and this concludes the proof. □

Now, it is well known that finding a maximum clique in a graph is an NP-hard problem [14]. However, if the graph is a complete rainbow $[c]$-splittable graph then we can find the maximum monochromatic clique (monochromatic of a color in $[c]$) in polynomial time as proved by the following lemma:

Lemma 3. *For any rainbow $[c]$-splittable edge-colored complete graph K_n and a color $a \in [c]$, there is an algorithm that in time $O(n^{4c-4})$ outputs a maximum monochromatic clique of color a.*

Proof. The algorithm we provide will quite straightforwardly follow by the next claim:

Claim 1. *Let K be an edge-colored $[c]$-splittable complete graph. For any color $a \in [c]$ and every monochromatic clique C of color a there exists a maximum monochromatic clique C_{max}, of the same color, for which $|C \setminus C_{max}| \leq 2c - 3$.*

Proof. Given an edge-colored $[c]$-splittable complete graph K, pick a color $a \in [c]$ and let and C be an arbitrary monochromatic clique of color a. As K is $[c]$-splittable let $V_1, \ldots, V_c$ be a rainbow $[c]$-split of K. Consider the class V_a and observe that for any monochromatic clique C' of color a we have $|C' \setminus V_a| \leq c - 1$ as otherwise we will have an edge of color a in some set V_i with $i \neq a$. Now, if V_a is a maximum clique then we are done. Otherwise, let C_{max} be the maximum monochromatic clique of color a in K for which the value of $|C \setminus C_{max}|$ is minimal. Now, suppose on the contrary that $|C \setminus C_{max}| > 2c - 3$. Then there are at least $c - 1$ elements of V_a that do not belong to C_{max}, i.e. $|V_a \setminus C_{max}| \geq c - 1$. Observe that

$$|V_a| = |V_a \cap C_{max}| + |V_a \setminus C_{max}| \geq |V_a \cap C_{max}| + c - 1$$

Algorithm *Split* $(G, [c])$

Require: An edge colored complete graph $G = (V, E)$ and a set $[c]$ of colors.
Ensure: Returns TRUE if G is $[c]$-splittable and FALSE otherwise.

if $c = 1$ **then**
    if G is a monochromatic clique **then**
        return TRUE
    else
        return FALSE
    end if
end if
$C \leftarrow MaxClique(G, c, c)$;
for all $X \subseteq C$ with $0 \le |X| \le c - 1$ **do**
    for all $Y \subseteq V(G) \setminus C$ with $0 \le |Y| \le |X|$ **do**
        $Z \leftarrow C \setminus (X \cup Y)$;
        if $G[Z]$ is monochromatic of color c **and** $Split(G[V \setminus Z], [c-1])$ **then**
            return TRUE
        end if
    end for
end for
return FALSE

Fig. 1. Algorithm *Split* $(G, [c])$.

and also

$$|C_{max}| = |V_a \cap C_{max}| + |C_{max}\ Va| \le |V_a \cap C_{max}| + c - 1$$

which leads to a contradictions as $|V_a| < |C_{max}|$. □

According to the previous claim we can think of an algorithm to produce the maximum monochromatic clique of some given color a based on a local search. It starts from a monochromatic clique C consisting of a single vertex and at each step tries to increment the size of C by removing at most $k \le 2c - 3$ vertices and inserting $k + 1$ vertices. It can be easily seen that each step has time complexity $O\left(\binom{n}{2c-3}\binom{n}{2c-2}\right)$ and as the number of steps is $O(n)$ this algorithm requires $O(n^{4c-4})$ time. And this concludes the lemma. □

Remark. Note that the bound provided by Claim 1 cannot be improved. Indeed, consider the edge-colored complete graph K composed by 4 disjoint monochromatic cliques C_1, C_2, C_3, C_4 each of the same color color c and with $|C_1| = |C_3| = |C_4| = c - 1$ and $|C_2| = c - 2$. Let the edges between C_1 and C_2, C_2 and C_4 and C_4 and C_3 be all of color c and let the edges between C_2 and C_3, and C_1 and C_4 be all of color $c - 1$. Finally we define the color of the edges between C_1 and C_3 as follows: Let $v_1, \dots, v_{c-1}$ be the vertices of C_1 or any $i \in [c - 1]$ color by i all the edges incident to the vertex v_i.

Observe that K is $[c]$-splittable as for instance $C_2 \cup C_4$ induces a monochromatic clique of color c and the vertices of C_1 and C_3 can be split into $c - 1$ monochromatic edges of pairwise distinct colors in $[c - 1]$. At this point $C = C_1 \cup C_2$ is a monochromatic clique of color c of size $2c - 3$ and clearly $C_{max} = C_3 \cup C_4$ is the only maximum clique of color c with size $2c - 2$. Thus, we conclude by observing that $|C \setminus C_{max}| = |C| = 2c - 3$.
Consider now Algorithm Split presented in Fig. 1. We claim that this algorithm correctly determines if a complete edge colored graph is rainbow $[c]$-splittable. Observe first that we make use of the algorithm $MaxClique(G, c, a)$ which refers to the one explained in Lemma 3 that returns a monochromatic clique of color a which is of maximum cardinality if the graph is rainbow $[c]$-splittable.

Theorem 2. *For every constant c and complete edge colored graph K_n, Algorithm Split presented in Fig. 1 decides in time $O(n^{c^2+1})$ if K_n is rainbow $[c]$-splittable.*

Proof. First we prove that this algorithm returns true if and only if the complete edge colored graph K_n is rainbow $[c]$-splittable. The proof follows by induction on c and the base case $c = 1$ trivially holds. Now, suppose it holds for any value till $c - 1$ and consider the case we want to decide if K_n is rainbow $[c]$-splittable. There are two cases to consider: If K_n is $[c]$-splittable then from Lemma 3 we have that C is the maximum monochromatic clique of color c in K_n thus applying Lemma 2 and using the induction hypothesis the algorithm will return true. Otherwise, if K_n is not rainbow $[c]$-splittable then again from Lemma 2 there is no way to find a set Z as required in the lemma for which $K_n[Z]$ is monochromatic of color c and there is a rainbow $[c]$-split for $K_n[V \setminus Z]$ with $V_c = \emptyset$, and thus the algorithm will return false.

Now, fix a constant c and let $T(n, c)$ be the time complexity of Algorithm Split. We can prove that $T(n, c) = O(n^{c^2+1})$ by induction on the value of c. It is clear that $T(n, 1) = O(n^2)$ so the base case holds. On the other hand it is not difficult to see

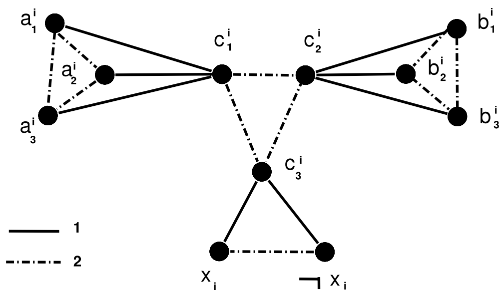

Fig. 2. The truth setting component for the variable x_i.

that the following holds

$$\begin{aligned} T(n,c) &= O(n^{4c-4}) + O\left(\binom{n}{c-1}^2 T(n,c-1)\right) \\ &= O(n^{4c-4}) + O(n^{2c-2+(c-1)^2+1}) \\ &= O(n^{c^2+1}). \end{aligned}$$

This concludes the proof of Theorem 2. □

Finally, if r is the number of colors that appear on the edges of a given complete graph then simply using Algorithm Split over all $\binom{r}{c}$ sets of c colors, we obtain the following:

Theorem 3. *For every constant c and r-colored complete graph K_n, there is an algorithm that decides in time $O(r^c n^{c^2+1})$ if K_n is rainbow c-splittable.*

4. Rainbow c-Splitting

In this section we prove that for graphs that are not necessarily complete the Rainbow c-Splitting problem is NP-complete for $c \geq 3$. Furthermore, it remains so, even if the graph has maximum degree $2c - 1$ and is c-colored. One may think that limiting the number of colors on the edges of the graph, changes the complexity of the problem. For instance, we trivially have that monochromatic graphs are rainbow c-splittable for every c. To this purpose, in Section 4.1 we focus on 2-colored graphs. We prove that even in this case the Rainbow c-Splitting problem is NP-complete for $c \geq 3$. Obviously this implies that the problem remains difficult even if more colors are allowed.

First we prove that Rainbow 3-Splitting Problem is NP-complete even for 3-colored graphs of maximum degree 5. The reduction is obtained using a restricted version of SAT problem defined as follows:

Definition 1 ((3, 2)-*SAT*)**.** Instance: A Boolean formula F in conjunctive normal form where each clause has two or three literals and each literal appears in at most 2 clauses.
Question: Is there some assignment of true and false value that will make the formula F true?

It is known that (3, 2)-SAT is NP-complete [18]. We use this result to prove the following:

Lemma 4. *Rainbow* 3*-Splitting Problem is NP-complete and it remains so even if the graph is* 3*-colored and has maximum degree* 5*.*

Proof. Let F be an instance of (3, 2)-SAT. To prove the lemma we show that it is possible to construct in polynomial time a 3-colored graph $G(F)$, with maximum degree 5 such that F is satisfiable if and only if $G(F)$ is rainbow 3-splittable. For the sake of simplicity we assume in this proof that the set of edge colors is $\{0, 1, 2\}$.

Assume that F has n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$. To each variable x_i, $1 \leq i \leq n$, we associate a *truth setting component* shown in Fig. 2.

The gadget has degree 5 and the two nodes labeled x_i and $\neg x_i$ are called literal-vertices of the component. It is easy to see that for any rainbow 3-split V_0, V_1, V_2 of the component at least one of the vertices in $\{a_1^i, a_2^i, a_3^i\}$ and at least one of the vertices in $\{b_1^i, b_2^i, b_3^i\}$ belong to set V_2. This implies that c_1^i must belong to one from the sets V_0, V_1 and the same is true for c_2^i. Due to the monochromatic triangle c_1^i, c_2^i, c_3^i this implies that c_3^i belongs to V_2. Finally, due to the triangle $x_i, c_3^i, \neg x_i$ it holds that x_i and $\neg x_i$ belong to different sets from V_0, V_1. Thus we have

Property 1. *For any rainbow* 3*-split V_0, V_1, V_2 of a truth setting component one of the two literal-vertices belongs to V_0 and the other to V_1.*

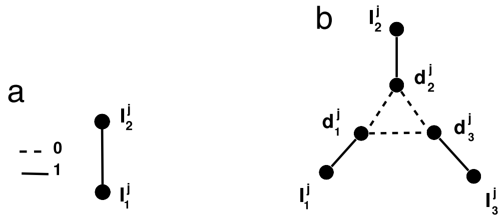

Fig. 3. (a) The test 2-component for the clause $C_j = \{l_1^j, l_2^j\}$. (b) The test 3-component for the clause $C_j = \{l_1^j, l_2^j, l_3^j\}$.

To each clause with two literals $C_j = \{l_1^j, l_2^j\}$, $1 \le j \le m$, we associate a *test* 2-*component* shown in Fig. 3(a). The two vertices of the component are said literal-vertices of the component.
To each clause with three literals $C_j = \{l_1^j, l_2^j, l_3^j\}$, $1 \le j \le m$, we associate a *test* 3-*component* shown in Fig. 3(b). The gadget has degree 3 and again the three vertices labeled l_1^j, l_2^j and l_3^j are said literal-vertices of the component. Note that in any rainbow 3-split V_0, V_1, V_2 of the component, due to the monochromatic triangle d_1^j, d_2^j, d_3^j, at least one of the vertices d_1^j, d_2^j and d_3^j belongs to V_0. This implies that at least one of the three literal-vertices of the component does not belong to V_0. Thus we have

Property 2. *For any rainbow* 3*-split* V_0, V_1, V_2 *of a test* 2*-component (test* 3*-component) whose literal-vertices belong to classes* V_0 *and* V_1*, at least one literal-vertex belongs to* V_1*.*

Truth setting components and test components are connected by identifying the literal-vertices corresponding to the same literal. In this fashion we obtain a 3-colored graph $G(F)$ of maximum degree 5 (remember that each literal occurs at most 2 times in F hence the degree of a literal-vertex in $G(F)$ has degree at most 4).
It remains to show that F is satisfiable if and only if $G(F)$ is rainbow 3-splittable.

- *the if part.* Consider a truth assignment that satisfies the formula F. We partition the literal-vertices of $G(F)$ in the sets V_0 and V_1 according to the assignment (i.e. we put the literal-vertex x_i in V_1 (resp. V_0) and vertex $\neg x_i$ in V_0 (resp. V_1) if x_i is true (resp. false) in the assignment. We can complete the partition of the vertices in the truth setting components inserting vertices $\{a_1^i, a_2^i, a_3^i, b_1^i, b_2^i, b_3^i, c_3^i\}$ in V_2, vertex c_1^i in V_0 and vertex c_2^i in V_2, $1 \le i \le n$. Finally, we complete the partition for the vertices in the test 3-components assigning for every $1 \le j \le m$ the three vertices $\{d_1^j, d_2^j, d_3^j\}$ to V_0 if all the literal vertices $\{l_1^j, l_2^j, l_3^j\}$ belong to set V_1 or otherwise, inserting them in three different sets (remember that at least one of the literal vertices belongs to set V_1 since the assignment satisfies the formula F). Thus the graph $G(F)$ is rainbow 3-splittable.
- *the only if part.* Consider a rainbow 3-split V_0, V_1, V_2 of the graph $G(F)$. This induces a rainbow 3-split for each truth setting component and by Property 1 each literal-vertex x_i belongs to V_0 or V_1. We can consider this partition as a truth assignment of the n variables of the formula F. Since the rainbow 3-split of $G(F)$ implies also a rainbow 3-split of all the test components, we have, by Property 2, that in this assignment at least one literal is satisfied for each clause of the formula. Thus, the formula is satisfiable.

This completes the proof. □

We are now ready to consider the Rainbow c-Splitting problem, for an arbitrary but fixed $c \ge 3$.

Theorem 4. *The Rainbow c-Splitting problem is NP-complete for any fixed integer* $c \ge 3$ *and remains so, even if the graph is* c*-colored and has maximum degree* $2c - 1$*.*

Proof. We prove the theorem by induction on $c \ge 3$. Observe that the base case is true due to Lemma 4. For the inductive step we show how to construct in polynomial time from a $(c-1)$-colored graph G with maximum degree $2c-3$ a c-colored graph G' with maximum degree $2c-1$ such that G is rainbow $(c-1)$-splittable if and only if G' is rainbow c-splittable.

Consider the gadget H_c in Fig. 4 consisting of c^2 vertices connected as follows: A vertex a is connected to all the vertices $\{b_1, b_2, \ldots, b_{c-1}\}$ with edges of color c. The vertices $\{b_1, b_2, \ldots, b_{c-1}\}$ form a monochromatic clique of color c. Each vertex b_i, $1 \le i \le c-1$, is connected to all the vertices of a clique D_i with edges of color 1. Finally D_i is a monochromatic clique of color c and size c.

Obviously H_c has maximum degree $2c-1$. Moreover for any rainbow c-split $V_1, \ldots, V_c$ of H_c at least one of the vertices of the clique D_i is in V_c. This implies that none from the vertices in $\{b_1, b_2, \ldots, b_{c-1}\}$ may belong to V_c and, since these vertices form a monochromatic clique of color c, they must be in different sets of the partition. Hence, vertex a can not be in any of the $c-1$ sets $V_1, V_2, \ldots, V_{c-1}$ and consequently it must belong to set V_c. Thus, the gadget H_c has the following property

Property 3. *Any rainbow* c*-split* $V_1, \ldots, V_c$ *of the gadget* H *would put vertex* a *in set* V_c*.*

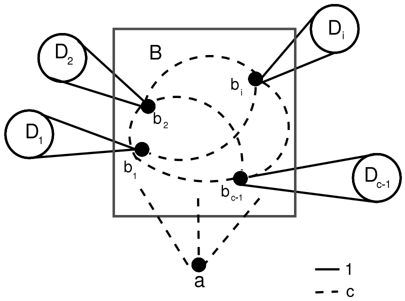

Fig. 4. The gadget H_c where B is a monochromatic clique of size $c-1$ and color c and D_i with $1 \le i \le c-1$ is a monochromatic clique of size c and color c.

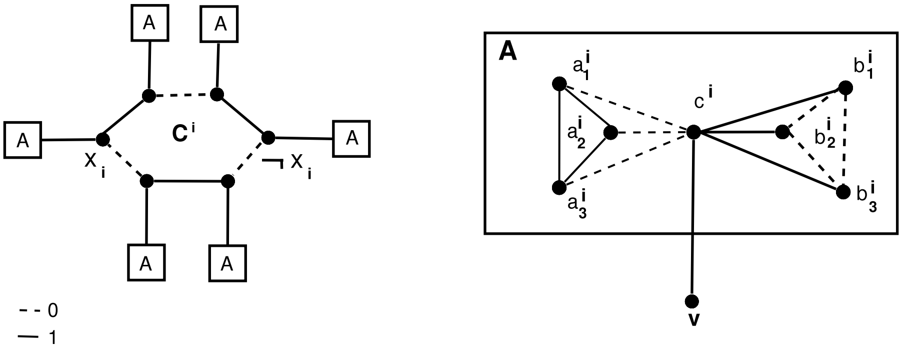

Fig. 5. The truth setting component for the variable x_i.

The graph G' is constructed from graph G adding a gadget H_c for each vertex x in G and connecting the vertex a of the gadget to the vertex x with an edge of color 1. Since the degree of the gadget is $2c-1$, vertex a in the gadget has degree $c-1$ and any vertex in G has degree at most $2c-3$ we have that G' has maximum degree $2c-1$. Moreover, due to Property 3 in any rainbow c-split of G', vertex a is in V_c and the edge $\{a, x\}$ colored by 1 implies that no vertex of the graph G is in set V_c. Thus any rainbow c-split of G' induces a rainbow $(c-1)$-split of G. Finally, observe that as the color of the edges of G belong to $[c-1]$, it has no split set inducing a monochromatic clique of color c. In view of this, it is not difficult to see that from any rainbow $(c-1)$-split of G we can obtain a rainbow c-split of G'. This completes the proof. □

4.1. *Rainbow c-splitting of 2-colored graphs*

Here we prove that the Rainbow c-Splitting problem is NP-complete even if G is 2-colored. We first consider the case $c = 3$. Thus, we define the following problem:

Rainbow (3, 2)-Splitting Problem
Instance: A 2-colored graph G.
Question: Is G rainbow 3-splittable?
For the sake of simplicity, assume that the two colors appearing on the edges are 0 and 1.

Lemma 5. *Rainbow* $(3, 2)$*-Splitting problem is NP-complete even if the graph has maximum degree* 7*.*

Proof. The proof follows the same lines of the proof of Lemma 4. The only difference is in the truth setting component that we assign to each variable x_i, $1 \le i \le n$. The new gadget, shown in Fig. 5, has degree 7 (due to the vertex c^i in the subgraph A). It is easy to see that for any rainbow 3-split V_0, V_1, V_2 of the component due to the monochromatic triangles of the subgraph A, at least one of the vertices in $\{a_1^i, a_2^i, a_3^i\}$ belongs to V_1 and at least one of the vertices in $\{b_1^i, b_2^i, b_3^i\}$ belongs to V_0. Thus, the vertex c^i belongs to V_2, deducing that each of the vertices of the cycle C^i belongs to V_0 or V_1. It is not difficult to see that as in the cycle C^i the color of the edges alternates, x_i and $\neg x_i$ belong to different sets from V_0, V_1. Thus, the following property holds.

Property 4. *For any rainbow* 3*-split* V_0, V_1, V_2 *of a truth setting component one of the two literal-vertices belongs to* V_0 *and the other to* V_1*.*

The test component associated to each clause remains the same as shown in Fig. 3. Using the same argument as in Lemma 4 concerning the test component, Property 2 holds.

It is not difficult to see that if F is an instance of (3, 2)-SAT, it is possible to construct in polynomial time a 2-colored graph $G(F)$, with maximum degree 7 such that F is satisfiable if and only if $G(F)$ is rainbow 3-splittable. Indeed, on one hand

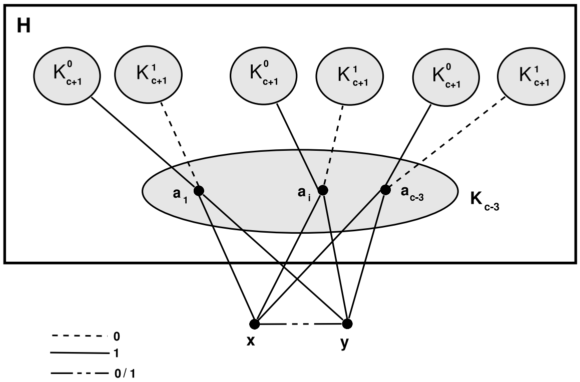

Fig. 6. The gadget H where K_{c-3} is a monochromatic clique of size $c-3$ and color 1 and K^0_{c+1} and K^1_{c+1} are monochromatic cliques of size $c+1$ and respectively of colors 0, 1.

for a truth assignment that satisfies the formula F, we partition the literal-vertices of $G(F)$ in the sets V_0 and V_1 according to the assignment. It is clear we can complete the partition of the vertices in the truth setting components inserting vertices a^i_1, a^i_2, a^i_3 in V_1, vertices b^i_1, b^i_2, b^i_3 in V_0, vertex c^i_1 in V_2, and assigning alternatively the vertices of the cycle C^i to the sets V_0 and V_1, for any $1 \leq i \leq n$. Finally, we complete the partition for the vertices in the test components in a similar way as in Lemma 4. Thus, the graph $G(F)$ has a rainbow 3-split.

On the other hand a rainbow 3-split V_0, V_1, V_2 of $G(F)$ induces a 3-partition for each truth setting component. By Property 4 each literal-vertex x_i belongs to V_0 or V_1. We can consider this partition as a truth assignment of the n variables of the formula F. Since the rainbow 3-split of $G(F)$ implies also a rainbow 3-split of all the test components, we have, by Property 2, that in this assignment at least one literal is satisfied for each clause of the formula. Thus the formula is satisfiable. This concludes the proof. □

Now, we are ready to consider the following problem:

Rainbow (c, 2)-Splitting Problem

Instance: A 2-colored graph G.

Question: Is G rainbow c-splittable?

Theorem 5. *The Rainbow $(c, 2)$-Splitting problem is NP-complete for any constant $c \geq 3$ even if the graph has maximum degree $7c - 14$.*

Proof. We show that Rainbow (3, 2)-Splitting problem is reducible to Rainbow $(c, 2)$-Splitting problem. To this purpose we show how to construct in polynomial time from a 2-colored graph G with maximum degree 7 a 2-colored graph G' with maximum degree $7c - 14$ such that G is rainbow 3-splittable if and only if G' is rainbow c-splittable.

To this purpose consider the gadget H in Fig. 6 consisting of a monochromatic clique K_{c-3} of color 1 where any vertex a_i of it (with $0 \leq i \leq c-3$) is connected to all the vertices of two monochromatic cliques K^0_{c+1} and K^1_{c+1} respectively of color 0 and 1. Furthermore, all the edges going to K^0_{c+1} are of color 1 and conversely all those going to K^1_{c+1} are of color 0. Now given a 2-colored graph G we construct G' as follows: For any edge $xy \in E(G)$ we insert a copy of the gadget H in the graph and connect x and y to all the vertices of K_{c-3} with edges of color 1, as shown in Fig. 4.

It is not difficult to see that the maximum degree of G' is $7c - 14$. Indeed, any vertex that belongs to one of the cliques K_{c+1} of a gadget is of degree $c+2$, any vertex a_i of K_{c-3} of a gadget (with $1 \leq i \leq c-3$) is of degree $c-4+2(c+1)+2$, and finally any vertex $x \in V(G)$ has $d_{G'}(x) = d_G(x) + d_G(x)(c-3) \leq 7c - 14$ as every vertex in G is of degree at most 7.

Moreover for any rainbow c-split $V_0, V_1, \ldots, V_{c-1}$ of H, at least two of the vertices of the cliques K^0_{c+1} (K^1_{c+1}) are in the same set inducing necessarily a subgraph of color 0 (1). Thus, any c-split of G' must have two sets V_0 and V_1 which induce two monochromatic subgraphs of colors 0 and 1 respectively. This implies that none of the vertices in $a_1, a_2, \ldots, a_{c-3}$ may be either in V_0 or in V_1 and no two of them belong to the same partition set of the split. Thus the following property holds:

Property 5. *Any rainbow c-split $V_0, V_1, \ldots, V_{c-1}$ of the gadget H has exactly one empty partition set.*

For any gadget H we denote this partition set by V^*_H. Now, observe that in any c-split of G', for any edge xy of G we have that x and y may belong only to one of the sets V_0, V_1, or V^*_H where H is the gadget in G' corresponding to the edge xy. Observing that x and y are also connected in G' by an edge of color 0 or 1 we have that if $x \notin V_0 \cup V_1$ then $y \in V_0 \cup V_1$. Thus the following property holds:

Property 6. *For any rainbow c-split $V_0, V_1, \ldots, V_{c-1}$ of the graph G' the set $\hat{V} = \cup_{i=2}^{c-1} V_i$ induces a stable set in G.*

It is clear now that if $V_0, V_1, \ldots, V_{c-1}$ is a rainbow c-split of G' then the sets $V_0 \cap V(G)$, $V_1 \cap V(G)$ and $\hat{V} \cap V(G)$ form a rainbow 3-split of G. Conversely, it is not difficult to see that any rainbow 3-split of G can be extended to a rainbow c-split $V_0, V_1, \ldots, V_{c-1}$ of G' by assigning to the split sets the vertices of any of the copies of the gadget H as follows: The vertices of any of the cliques K^0_{c+1} (K^1_{c+1}) belong to the set V_0 (V_1) and for $1 \leq i \leq c-3$, a_i belongs to the set V_{i+3}. This completes the proof. □

5. Conclusions and open problems

In this paper we approached the problem of recognizing rainbow c-splittable graphs, for some fixed integer c. First, we showed that rainbow 2-splittable graphs can be recognized in polynomial time by providing a reduction to the well-known 2-SAT problem. Next, we focused on the case of complete graphs for which we proposed a polynomial-time algorithm that solves the Rainbow c-Splitting problem. For not necessarily complete graphs and $c \geq 3$ we proved that the problem is NP-complete even if the graph is c-colored and has maximum degree $2c - 1$.

Observe that Brooks' theorem [4] states that every connected non-complete graph G of maximum degree $c \geq 3$ has a chromatic number of at most c. Thus, it is clear that edge-colored graphs of maximum degree $c \geq 3$ are trivially rainbow c-splittable (note that K_{c+1} is trivially rainbow c-splittable). However, in spite of our efforts, the following problem remains unanswered.

Problem 1. What is the computational complexity of the Rainbow c-Splitting problem for edge-colored graphs of maximum degree between $c + 1$ and $2c - 2$?

We also showed that the Rainbow c-Splitting problem in case of 2-colored graphs is still NP-complete, even if the graph has maximum degree $7c - 14$. It seems reasonable to think that some relation must hold between the number of colors on the edges of the graph and the maximum degree for which the rainbow c-splittance can be decided in polynomial time. Therefore, investigating in this direction may be also interesting.

In this paper we considered only the case when c is fixed, although it should be clear that the problem makes sense for non-fixed c. It should be interesting to study the rainbow c-splittance when c is part of the input. In particular, observe that the polynomial-time algorithm that decides rainbow c-splittance of complete graphs, heavily relies on the fact that c is a fixed constant. Thus, it would be nice to know:

Problem 2. If c is not fixed (i.e. c is part of the input), can rainbow c-splittance of complete graphs be still decided in polynomial time?

We conclude this section with a further observation. Starting from the notion of adaptable coloring, Hell and Zhu [12] introduced the notion of *adaptable chromatic number*.

Definition 2 (*[12]*)**.** The adaptable chromatic number of a simple (non-colored) graph G, denoted $\chi_a(G)$, is the minimum integer k such that for every edge-coloring f of G using k colors there exists a coloring $\hat{f}$ of vertices using the same k colors such that for every edge xy, $\hat{f}(x) = \hat{f}(y) = a$ implies $f(xy) \neq a$.

In the same flavor we can define a similar concept considering the rainbow splitting as a vertex coloring problem.

Definition 3. The *relaxed chromatic number* of a simple (non-colored) graph G, denoted $\chi_R(G)$, is the minimum integer k such that for every edge-coloring f of G there exists a coloring $\hat{f}$ of vertices using k colors such that for every edge xy, $\hat{f}(x) = \hat{f}(y) = a$ implies $f(xy) = a$.

It is not difficult to prove that for any graph G the following holds:

$$\chi_a(G) \leq \chi_R(G) \leq \chi(G).$$

We believe that this new concept of chromatic number is of independent interest and paves the way for many interesting and challenging problems. For instance it is not difficult to prove that determining the relaxed chromatic number of an arbitrary graph is an NP-hard problem and in [12] it is proved that the same is true also for the adaptable chromatic number. Another interesting problem is to determine the relaxed chromatic number for complete graphs. It is clear that $\chi_R(K_n) < n$. Moreover, we have an edge-coloring of K_n from which we deduce $\chi_R(K_n) \geq n - \log n$. For the adaptable chromatic number of complete graphs the problem was first studied in [12] and is completely solved in the asymptotic in [17] where it is proved that $\chi_a(K_n) = (1 + o(1))\sqrt{n}$.

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